Published by Institute of Physics Publishing for SISSA

Received: *June 18, 2007*
Revised: *August 20, 2007*
Accepted: *October 10, 2007*
Published: *October 19, 2007*

The Hitchin model, Poisson-quasi-Nijenhuis, geometry and symmetry reduction

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Abstract: We revisit our earlier work on the AKSZ-like formulation of topological sigma model on generalized complex manifolds, or Hitchin model, [20]. We show that the target space geometry geometry implied by the BV master equations is Poisson-quasi-Nijenhuis geometry recently introduced and studied by Stiénon and Xu (in the untwisted case) in [44]. Poisson-quasi-Nijenhuis geometry is more general than generalized complex geometry and comprises it as a particular case. Next, we show how gauging and reduction can be implemented in the Hitchin model. We find that the geometry resulting form the BV master equation is closely related to but more general than that recently described by Lin and Tolman in [40, 41], suggesting a natural framework for the study of reduction of Poisson-quasi-Nijenhuis manifolds.

Keywords: Sigma Models, Topological Field Theories, BRST Symmetry, Differential and Algebraic Geometry.

Contents

1. Introduction

In type II superstring theory, an effective four dimensional low energy field theory is obtained by compactification of the six extra dimensions. In the absence of fluxes, requiring unbroken four dimensional $\mathcal{N} = 2$ supersymmetry leads to the well known condition that the six dimensional internal manifold should be Calabi-Yau. In recent years, a large body of literature has been devoted to attempts to find a similarly elegant condition in the presence of NS and RR fluxes, both for $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry. (See for instance [1] for a comprehensive review and extensive referencing). The intense scrutiny, which these more general compactifications have undergone, reflects both their physical and mathematical interest.

In flux compactifications of type II superstring theories, requiring unbroken four dimensional $\mathcal{N} = 1$ supersymmetry leads to certain topological and differential conditions on the internal manifold M [2−4]. These conditions are naturally expressed in the mathematical language of generalized complex geometry [5, 6]. (See [7−9] for recent reviews of this subject aimed to a physical readership). They state the existence of two nowhere vanishing globally defined $TM \oplus T^*M$ pure spinors. One of these satisfies the appropriate differential condition required for it to define a twisted generalized Calabi-Yau structure on M. The other, conversely does not, the obstruction being due to the presence of RR fluxes and warping.

Ordinary fluxless type II compactifications are described by $(2,2)$ superconformal sigma models on Calabi-Yau manifolds. These are however nonlinear interacting field theories and, so, are rather complicated and difficult to study. In 1988, Witten showed that a $(2,2)$ supersymmetric sigma model on a Calabi-Yau manifold could be twisted in two different ways, to yield the so called A and B topological sigma models [10, 11]. Unlike the original untwisted sigma models, the topological models are soluble field theories: the calculation of observables can be carried out by standard methods of geometry and topology.

The recent interest in flux compactifications has prompted the search for topological sigma models on generalized complex manifolds. In the particular case of biHermitian manifolds [12], this problem was tackled in [13, 14] by Kapustin and Kapustin and Li, who formulated it in the suitable geometrical framework of generalized Kaehler geometry [6] and derived the appropriate twisting prescriptions. In refs. [15 − 17], developing on Kapustin's and Li's results, the biHermitian topological action and symmetry variations were explicitly derived and written down.

BiHermitian geometry can accommodate only NS flux. If one wishes to incorporate RR fluxes, it is non longer sufficient. In the last few years, many attempts have been made to construct topological sigma models with generalized complex target manifolds more general than generalized Kaehler ones [18 − 22]. All these endeavors were somehow unsatisfactory either because they remained confined to the analysis of geometrical aspects of the sigma models or because they yielded field theories which were not directly suitable for quantization. In [20 − 22], the sigma models were constructed by employing the Batalin-Vilkovisky (BV) quantization algorithm [23, 24] in the Alexandrov-Kontsevich-Schwartz-Zaboronsky (AKSZ) formulation [25]. To date, this seems to be the most promising approach to the solution of the problem of constructing interesting sigma models on generalized complex target manifolds, though, as shown in [26], the implementation of gauge fixing remains a major technical obstacle even in the simplest cases.

One efficient way of generating sigma models on non trivial manifolds is the gauging of sigma models on simpler manifolds. The target space of the gauged model turns out to be the quotient of that of the ungauged model by an action of the gauge group. In certain cases, when a symplectic structure and a moment map for the gauge group action can be defined, this construction is a particular case of a general procedure called Hamiltonian reduction [27]. The gauging of (2,2) supersymmetric sigma models on biHermitian manifolds was studied originally by Hull, Papadopoulos and Spence in [28] developing on the results of [12]. Their analysis was however limited to the subclass of almost product structure biHermitian spaces because of the lack of an off-shell (2,2) supersymmetric action in the general case at that time. Recently, such action has been obtained in ref. [29]. This has led the authors of [30] to extend the analysis of [28] for general biHermitian target spaces. In [31], the same analysis has been carried out in the on-shell formalism. Simultaneously, many mathematical studies of the problem of reduction of generalized complex, Calabi-Yau and Kaehler manifolds have appeared [32 − 43], calling for a comparison with the target space geometries yielded by sigma model gauging.

In this paper, we revisit our earlier work on the AKSZ type formulation of topological sigma model on generalized complex manifolds, or Hitchin model, which we introduced in 2004 in [20]. We show that the target space geometry geometry encoded in the BV master equations is twisted Poisson-quasi-Nijenhuis geometry recently introduced and studied by Stiénon and Xu (in the untwisted case) in [44]. Poisson-quasi-Nijenhuis geometry is more general than generalized complex geometry and comprises it as a particular case. This should clarify the issue of the underlying geometry of the Hitchin model raised but not solved in [20]. Next, we show how gauging (here meant in a non standard way explained in the following) can be incorporated in the Hitchin model. We find that the geometry resulting form the BV master equation is closely related to but more general than that described by Lin and Tolman in [40, 41] and is fully b symmetry covariant, suggesting a natural framework for the study of reduction of twisted Poisson-quasi-Nijenhuis manifolds.

The plan of the paper is as follows. In section 2, we review the basic features of an AKSZ type formulation of topological sigma models relevant in the following. In section 3, we introduce the Weil sigma model, a canonical sigma model associated to any real Lie algebra, and study it in the AKSZ framework. In section 4, we review the AKSZ formulation of the Poisson sigma model and gauge it by coupling it to the Weil model. This introduces section 5, where we revisit the AKSZ formulation of the Hitchin sigma model showing that the underlying geometry is twisted Poisson-quasi-Nijenhuis and gauge it by coupling it again to the Weil model. In section 6, we study the geometry of the Hitchin-Weil model and show substantial evidence that this may encode a rather general reduction scheme for Poisson-quasi-Nijenhuis geometry. Finally, in section 7, we discuss the results obtained.

2. The AKSZ scheme

The Alexandrov-Kontsevich-Schwartz-Zaboronsky (AKSZ) formalism of ref. [25] is a method of constructing solutions of the Batalin-Vilkovisky (BV) classical master equation directly, without starting from a classical action with a set of symmetries, as is usually done in the BV framework [23, 24]. In ref. [45, 46], using such formalism, Cattaneo and Felder managed to obtain the BV action of the Poisson sigma model [47, 48]. Their approach is closely related the one of the present paper. For this reason, we shall review it briefly. We refer the reader to app. A for a review of de Rham superfield formalism used throughout this paper.

The AKSZ formulation of the Poisson sigma model of ref. [46] can be summarized in the following terms.

1. The field space is $\mathrm{Maps}(T[1]\Sigma, Y)$, where Σ is a closed surface and $Y = T^*[1]M$ with M a smooth manifold.

2. $T[1]\Sigma$ is endowed with a degree 1 homological vector field D and a D invariant measure ϱ of degree 2. Under the isomorphism $\mathrm{Fun}(T[1]\Sigma) \simeq \Omega^*(\Sigma)$, D corresponds to the de Rham differential and ϱ to the usual Lebesgue measure of Σ.

3. Y is endowed with a degree 1 symplectic form 2-form ω and a degree 1 homological vector field Q that is Hamiltonian with degree 2 Hamiltonian function $P \in \mathrm{Fun}(Y)$.

ω is the standard degree 1 symplectic form of Y and P is a degree 2 element of $\mathrm{Fun}(Y)$ corresponding to a Poisson 2-vector.

4. Using the above data, a degree -1 symplectic form $\tilde{\omega}$ on $\mathrm{Maps}(T[1]\Sigma, Y)$ is constructed by pulling back the symplectic form ω of Y by the evaluation map $\mathrm{ev}: T[1]\Sigma \times \mathrm{Maps}(T[1]\Sigma, Y) \to Y$ and integrating on $T[1]\Sigma$ with respect to the measure ϱ. This yields the BV symplectic form and the BV antibrackets $(\cdot,\cdot)$ of the Poisson sigma model.

5. The homological vector field D of $T[1]\Sigma$ induces a homological vector field $\hat{D}$ on $\mathrm{Maps}(T[1]\Sigma, Y)$. This is Hamiltonian with degree 0 Hamiltonian function S_1 satisfying $(S_1, S_1) = 0$.

6. The homological vector fields Q of Y induces a homological vector field $\tilde{Q}$ on $\mathrm{Maps}(T[1]\Sigma, Y)$. This is Hamiltonian with degree 0 Hamiltonian function S_2 satisfying $(S_2, S_2) = 0$. S_2 is obtained by composing P with the given field $\phi \in \mathrm{Maps}(T[1]\Sigma, Y)$ and integrating on $T[1]\Sigma$ with respect to the measure ϱ.

7. $\hat{D}$ and $\tilde{Q}$ anticommute. Correspondingly, $(S_1, S_2) = 0$.

8. The Poisson sigma model action S is just $S = S_1 + S_2$.

In this paper, we consider sigma models with the following general features.

1. The field space is of the form $\mathrm{Maps}(T[1]\Sigma, Y)$, where Σ is again a closed surface and Y is a smooth supermanifold (roughly a grade shifted cotangent bundle of some graded manifold N).

2. $T[1]\Sigma$ is endowed with a degree 1 homological vector field D and a D invariant measure ϱ of degree 2, the same as those of the Poisson sigma model.

3. Y is endowed with a degree 1 symplectic form 2-form ω and a degree 1 homological vector field Q that is Hamiltonian with degree 2 Hamiltonian function $P \in \mathrm{Fun}(Y)$. However, in the models treated in this paper, there are other relevant target space data which are not elements of $\mathrm{Fun}(Y)$.

4. A degree -1 symplectic form $\tilde{\omega}$ on $\mathrm{Maps}(T[1]\Sigma, Y)$ is constructed. In general, this contains a term obtained by pulling back the symplectic form ω of Y by the evaluation map $\mathrm{ev}: T[1]\Sigma \times \mathrm{Maps}(T[1]\Sigma, Y) \to Y$ and integrating on $T[1]\Sigma$, as in the Poisson sigma model, plus a twist term which does not originate this way. This yields anyway the BV symplectic form and the BV antibrackets $(\cdot,\cdot)$ of the sigma model.

5. The homological vector fields D of X induces a homological vector field $\hat{D}$ on $\mathrm{Maps}(T[1]\Sigma, Y)$, that is Hamiltonian with degree 0 Hamiltonian function S_1 satisfying $(S_1, S_1) = 0$, as in the Poisson sigma model.

6. A homological vector field $\tilde{Q}$ on $\mathrm{Maps}(T[1]\Sigma, Y)$ is constructed. This is Hamiltonian with degree 0 Hamiltonian function S_2 satisfying $(S_2, S_2) = 0$. In general, S_2 is fully obtained from target space data and involves integration with respect to the measure ϱ but, however, it is not obtained by composing some element of $\mathrm{Fun}(Y)$ with the given field $\phi \in \mathrm{Maps}(T[1]\Sigma, Y)$: there are terms in S_2, which cannot cannot be generated in this way.

7. $\hat{D}$ and $\tilde{Q}$ anticommute. Correspondingly, $(S_1, S_2) = 0$, just as in the Poisson sigma model.

8. The sigma model action S is $S = S_1 + S_2$, just as in the Poisson case.

The above discussion indicates that, in spirit, the field theoretic geometrical scheme of the present paper is essentially the same as that of the AKSZ formulation of the Poisson sigma model of ref. [46]. For this reason, we shall it the AKSZ scheme, though, strictly speaking, it differs from the standard AKSZ scheme at significant points. The original AKSZ formulation, geometrically beautiful as it is, it is quite rigid and hardly allows any other model than the Poisson sigma model in 2 dimensions, as rightfully claimed Roytenberg in [49].

Our formal framework has the following basic features. Each sigma model, which we shall consider below, is characterized by a pair of action functionals S_r, $r = 1,\ 2$, which satisfy the joined BV master equation

$$(S_r, S_s) = 0, \qquad r,\ s = 1,\ 2. \tag{2.1}$$

With the S_r there are associated odd BV variations by

$$\delta_r \phi = (S_r, \phi), \tag{2.2}$$

where ϕ is any field of the model. When (2.1) holds, one has

$$\delta_r \delta_s + \delta_s \delta_r = 0, \qquad r,\ s = 1,\ 2. \tag{2.3}$$

The choice of the action functionals S_r, $r = 1,\ 2$, is non unique. One is allowed to carry out a linear redefinition of the form

$$S'_r = \sum_{s=1}^{2} A_{rs} S_s, \tag{2.4}$$

where $(A_{rs})_{r,s=1,2}$ is a non singular 2 by 2 matrix. For each sigma model considered in this paper, it is possible to choose S_1 in such a way that for any field ϕ,

$$\delta_1 \phi = d\phi. \tag{2.5}$$

where d is the de Rham differential of the world sheet. Thus, δ_1 is nothing but the homological vector field $\hat{D}$ discussed above

$$\delta_1 = \hat{D}. \tag{2.6}$$

δ_2 is then identified with the homological vector field $\tilde{Q}$.

$$\delta_2 = \tilde{Q}. \tag{2.7}$$

Since S_2 is defined up to the addition of a multiple of S_1, $\tilde{Q}$ is correspondingly defined up to the same multiple of $\hat{D}$. One may try to use this freedom to make S_2 and so $\tilde{Q}$ to be related in some meaningful way to the geometry of Y, as in the Poisson sigma model, but, as already noticed, this cannot be achieved in general.

3. The Weil sigma model

In this section, we introduce the Weil sigma model, which plays an important role in the following. The Weil model is a canonical sigma model associated to any real Lie algebra $\mathfrak{g}$. As it will turn out, coupling to the Weil model implements the gauging of the symmetry associated with the connected Lie group G having $\mathfrak{g}$ as its Lie algebra.

The field content of the model consists of fields $\beta \in C^\infty(T[1]\Sigma, \mathfrak{g}^\vee[0])$, $\gamma \in C^\infty(T[1]\Sigma, \mathfrak{g}[1])$, $\mathrm{B} \in C^\infty(T[1]\Sigma, \mathfrak{g}^\vee[-1])$ and $\Gamma \in C^\infty(T[1]\Sigma, \mathfrak{g}[2])$, where $\mathfrak{g}$ is for the time being a real vector space. The BV odd symplectic form is

$$\Omega_W = \int_{T[1]\Sigma} \varrho \Big[\delta\beta_i \delta\gamma^i + \delta \mathrm{B}_i \delta\Gamma^i\Big]. \tag{3.1}$$

This satisfies obviously

$$\delta\Omega_W = 0. \tag{3.2}$$

The associated BV rackets are

$$(F, G)_W = \int_{T[1]\Sigma} \varrho \Big[\frac{\delta_r F}{\delta\beta_i}\frac{\delta_l G}{\delta\gamma^i} - \frac{\delta_r F}{\delta\gamma^i}\frac{\delta_l G}{\delta\beta_i} + \frac{\delta_r F}{\delta \mathrm{B}_i}\frac{\delta_l G}{\delta\Gamma^i} - \frac{\delta_r F}{\delta\Gamma^i}\frac{\delta_l G}{\delta \mathrm{B}_i}\Big], \tag{3.3}$$

for any two functionals F, G on field space.

The model is characterized by two basic action functionals given by

$$S_{W1} = \int_{T[1]\Sigma} \varrho \Big[\beta_i d\gamma^i - \mathrm{B}_i d\Gamma^i\Big], \tag{3.4a}$$

$$S_{W2} = \int_{T[1]\Sigma} \varrho \Big[\beta_i \Gamma^i - \frac{1}{2} f^i{}_{jk}\beta_i\gamma^j\gamma^k - f^i{}_{jk}\mathrm{B}_i\Gamma^j\gamma^k\Big], \tag{3.4b}$$

where $f \in \mathfrak{g} \otimes \wedge^2\mathfrak{g}^\vee$. A simple computation yields the BV brackets

$$(S_{W1}, S_{W1})_W = 0, \tag{3.5a}$$

$$(S_{W1}, S_{W2})_W = 0, \tag{3.5b}$$

$$(S_{W2}, S_{W2})_W = 2\int_{T[1]\Sigma} \varrho \Big[\frac{1}{6} g^i{}_{jkl}\beta_i\gamma^j\gamma^k\gamma^l + \frac{1}{2} g^i{}_{jkl}\mathrm{B}_i\Gamma^j\gamma^k\gamma^l\Big], \tag{3.5c}$$

where $g \in \mathfrak{g} \otimes \wedge^3\mathfrak{g}^\vee$ is given by

$$g^i{}_{jkl} = f^i{}_{mj} f^m{}_{kl} + f^i{}_{mk} f^m{}_{lj} + f^i{}_{ml} f^m{}_{jk}. \tag{3.6}$$

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Therefore, the joined BV master equations

$$(S_{Wr}, S_{Ws})_W = 0, \qquad r,\ s = 1,\ 2, \tag{3.7}$$

are satisfied if and only if

$$g^i{}_{jkl} = 0, \tag{3.8}$$

that is when $\mathfrak{g}$ is a Lie algebra with structure constants $f^i{}_{jk}$.

The BV variations associated with the actions S_{Wr} are defined according to (2.2) as $\delta_{Wr} = (S_{Wr},\ )_W$. Explicitly,

$$\delta_{W1}\beta_i = d\beta_i, \tag{3.9a}$$

$$\delta_{W1}\gamma^i = d\gamma^i, \tag{3.9b}$$

$$\delta_{W1}\mathrm{B}_i = d\mathrm{B}_i, \tag{3.9c}$$

$$\delta_{W1}\Gamma^i = d\Gamma^i, \tag{3.9d}$$

$$\delta_{W2}\beta_i = -f^j{}_{ik}\beta_j\gamma^k - f^j{}_{ik}\mathrm{B}_j\Gamma^k, \tag{3.9e}$$

$$\delta_{W2}\gamma^i = \Gamma^i - \frac{1}{2}f^i{}_{jk}\gamma^j\gamma^k, \tag{3.9f}$$

$$\delta_{W2}\mathrm{B}_i = -\beta_i + f^j{}_{ik}\mathrm{B}_j\gamma^k, \tag{3.9g}$$

$$\delta_{W2}\Gamma^i = -f^i{}_{jk}\gamma^j\Gamma^k. \tag{3.9h}$$

From the above analysis, it follows that the Weil sigma model can be framed in the AKSZ scheme of section 2.

The Weil sigma model fields β, γ, B, Γ define together a map of $T[1]\Sigma$ into $T^*[1]\mathfrak{g}^\vee[0]\oplus T^*[3]\mathfrak{g}^\vee[-1]$. Hence, the Weil sigma model can be viewed as a Poisson sigma model whose target space is the graded vector space $\mathfrak{g}^\vee[0]\oplus\mathfrak{g}^\vee[-1]$. Albeit interesting, we shall not pursue this line of interpretation any further.

To any Lie algebra $\mathfrak{g}$, there is canonically associated the Weil algebra $W(\mathfrak{g}) = \wedge^*\mathfrak{g}^\vee[1]\otimes\vee^*\mathfrak{g}^\vee[2]$. This is the tensor product of the antisymmetric and symmetric algebras of $\mathfrak{g}^\vee$ in degree 1 and 2, respectively. The natural $\mathfrak{g}$-valued generators ω, Ω of $W(\mathfrak{g})$ carry degrees 1, 2, respectively. The Weil operator d_W acts as

$$d_W\omega^i = \Omega^i - \frac{1}{2}f^i{}_{jk}\omega^j\omega^k, \tag{3.10a}$$

$$d_W\Omega^i = -f^i{}_{jk}\omega^j\Omega^k, \tag{3.10b}$$

and is extended on $W(\mathfrak{g})$ by linearity. d_W is nilpotent

$$d_W{}^2 = 0. \tag{3.11}$$

The cohomology of $(W(\mathfrak{g}), d_W)$ is actually trivial.[1] It appears that the fields γ, Γ describe the embedding of $T[1]\Sigma$ into the Weil algebra. Further, by (3.9f), (3.9h), for any point

[1] As is well known, it is possible to define also a $\mathfrak{g}$ basic cohomology of $(W(\mathfrak{g}), d_W)$, which turns out to be non trivial.

$z \in T[1]\Sigma$, the evaluation map $\mathrm{e}_z : C^\infty(T[1]\Sigma, W(\mathfrak{g})) \mapsto W(\mathfrak{g})$ is a chain map of the chain complexes $(C^\infty(T[1]\Sigma, W(\mathfrak{g})), \delta_{W2})$, $(W(\mathfrak{g}), d_W)$. This justifies the name given to the sigma model considered above.

The Weil sigma model describes a supersymmetric gauge ghost system. The algebraic structure presented here is closely related to those appearing in the so called topological field theories of cohomological type. (See sect 10.3 of ref. [50] for a thorough review of these matters with many illustrative examples).

4. The Poisson-Weil sigma model

In this section, we illustrate the Poisson-Weil sigma model. This is interesting on its own and serves also the purpose of introducing the treatment of the more complicated Hitchin-Weil model expounded later. Our presentation is closely related to that of ref. [20], in turn inspired by refs. [45, 46].

The field content of the Poisson sigma model consists of a degree 0 embedding $x \in C^\infty(T[1]\Sigma, M)$ and a degree 1 section $y \in C^\infty(T[1]\Sigma, x^*T^*[1]M)$. The BV odd symplectic form is

$$\Omega_M = \int_{T[1]\Sigma} \varrho\, \delta x^a \delta y_a. \tag{4.1}$$

This satisfies obviously

$$\delta\Omega_M = 0. \tag{4.2}$$

The associated BV antibrackets are given by

$$(F, G)_M = \int_{T[1]\Sigma} \varrho \left[\frac{\delta_r F}{\delta x^a}\frac{\delta_l G}{\delta y_a} - \frac{\delta_r F}{\delta y_a}\frac{\delta_l G}{\delta x^a}\right], \tag{4.3}$$

for any two functionals F, G on field space. See app. B for technical details.

The model is characterized by two action functionals

$$S_{P1} = \int_{T[1]\Sigma} \varrho\, y_a dx^a, \tag{4.4a}$$

$$S_{P2} = \int_{T[1]\Sigma} \varrho\, \frac{1}{2} P^{ab}(x) y_a y_b, \tag{4.4b}$$

where $P \in C^\infty(M, \wedge^2 TM)$ is a 2-vector defining an almost Poisson structure on M.

A simple computation yields the BV brackets

$$(S_{P1}, S_{P1})_M = 0, \tag{4.5a}$$

$$(S_{P1}, S_{P2})_M = 0, \tag{4.5b}$$

$$(S_{P2}, S_{P2})_M = 2\int_{T[1]\Sigma} \varrho \left[-\frac{1}{6}A^{abc}(x) y_a y_b y_c\right], \tag{4.5c}$$

where the 3-vector $A \in C^\infty(M, \wedge^3 TM)$ is given by

$$A^{abc} = P^{ad}\partial_d P^{bc} + P^{bd}\partial_d P^{ca} + P^{cd}\partial_d P^{ab}. \tag{4.6}$$

Therefore, the joined BV master equations

$$(S_{Pr}, S_{Ps})_M = 0, \qquad r,\ s = 1,\ 2, \tag{4.7}$$

are satisfied if and only if

$$A^{abc} = 0. \tag{4.8}$$

As is well-known, condition (4.8) ensures the almost Poisson structure P is actually a Poisson structure, so that M is a Poisson manifold.

The BV variations associated with the actions S_{Pr} are defined according to (2.2) as $\delta_{Pr} = (S_{Pr},\ )_M$. Explicitly, one has

$$\delta_{P1}x^a = dx^a, \tag{4.9a}$$

$$\delta_{P1}y_a = dy_a, \tag{4.9b}$$

$$\delta_{P2}x^a = P^{ab}(x)y_b, \tag{4.9c}$$

$$\delta_{P2}y_a = \frac{1}{2}\partial_a P^{bc}(x)y_b y_c. \tag{4.9d}$$

From the above considerations, we see that the Poisson sigma model can be framed in the AKSZ scheme of section 2, as was in any case evident from the AKSZ analysis of ref. [46]

One can couple the Poisson and the Weil sigma models to obtain the Poisson-Weil sigma model. The field space of Poisson-Weil model is simply the Cartesian product of those of the Poisson and Weil models. The BV odd symplectic form Ω_{MW} of the Poisson-Weil model is correspondingly the sum of those of the Poisson and Weil models, $\Omega_{MW} = \Omega_M + \Omega_W$. Consequently, the BV antibrackets $(\ ,\ )_{MW}$ are the sum of the BV antibrackets $(\ ,\ )_M$ and $(\ ,\ )_W$ given by (4.3), (3.3).

The Poisson-Weil model is characterized by two action functionals:

$$S_{PW1} = S_{P1} + S_{W1}, \tag{4.10a}$$

$$S_{PW2} = S_{P2} + S_{W2} + \int_{T[1]\Sigma} \varrho\Big[- u_i{}^a(x)\gamma^i y_a + \mu_i(x)\Gamma^i\Big], \tag{4.10b}$$

where $u \in C^\infty(M, \mathfrak{g}^\vee \otimes TM)$ and $\mu \in C^\infty(M, \mathfrak{g}^\vee)$ are a $\mathfrak{g}^\vee$-valued vector field and a $\mathfrak{g}^\vee$-valued scalar on M, respectively.

A straightforward computation yields the BV brackets

$$(S_{PW1}, S_{PW1})_{MW} = 0, \tag{4.11a}$$

$$(S_{PW1}, S_{PW2})_{MW} = 0, \tag{4.11b}$$

$$(S_{PW2}, S_{PW2})_{MW} = (S_{P2}, S_{P2})_M + (S_{W2}, S_{W2})_W \tag{4.11c}$$

$$+ 2\int_{T[1]\Sigma} \varrho\left[\frac{1}{2}X_i{}^{ab}(x)\gamma^i y_a y_b - \frac{1}{2}L_{ij}{}^a(x)\gamma^i\gamma^j y_a \right. \tag{4.11d}$$

$$\left. + N_{ij}(x)\gamma^i\Gamma^j - S_i{}^a(x)\Gamma^i y_a\right],$$

where the BV antibrackets $(S_{P2}, S_{P2})_M$, $(S_{W2}, S_{W2})_W$ are given by (4.5c), (3.5c), respectively, and $X \in C^\infty(M, \mathfrak{g}^\vee \otimes \wedge^2 TM)$, $L \in C^\infty(M, \wedge^2 \mathfrak{g}^\vee \otimes TM)$, $N \in C^\infty(M, \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee)$, $S \in C^\infty(M, \mathfrak{g}^\vee \otimes TM)$ are given by

$$X_i{}^{ab} = u_i{}^c \partial_c P^{ab} - \partial_c u_i{}^a P^{cb} - \partial_c u_i{}^b P^{ac}, \tag{4.12a}$$

$$L_{ij}{}^a = u_i{}^b \partial_b u_j{}^a - u_j{}^b \partial_b u_i{}^a - f^k{}_{ij} u_k{}^a, \tag{4.12b}$$

$$N_{ij} = u_i{}^b \partial_b \mu_j - f^k{}_{ij} \mu_k, \tag{4.12c}$$

$$S_i{}^a = u_i{}^a + P^{ab} \partial_b \mu_i. \tag{4.12d}$$

The joined BV master equations

$$(S_{PWr}, S_{PWs})_{MW} = 0, \qquad r,\ s = 1,\ 2, \tag{4.13}$$

are satisfied if and only if (4.8), the conditions

$$N_{ij} = 0, \tag{4.14a}$$

$$S_i{}^a = 0, \tag{4.14b}$$

and (3.8) are simultaneously fulfilled. Indeed, it is easy to see that, when u_i is given by (4.14b), one has

$$X_i{}^{ab} = A^{abc} \partial_c \mu_i, \tag{4.15a}$$

$$L_{ij}{}^a = A^{abc} \partial_b \mu_i \partial_c \mu_j - P^{ab} \partial_b N_{ij}, \tag{4.15b}$$

The geometry of M emerging here will be analyzed in greater detail in section 6. We anticipate that that M is a Poisson manifold carrying an infinitesimal action of the Lie algebra $\mathfrak{g}$ leaving the Poisson structure P invariant, the action being Hamiltonian with equivariant moment map μ. This geometrical set up allows for the symmetry reduction of M, which is therefore encoded in the Poisson-Weil model.

The BV variations associated with the actions S_{PWr} are defined as usual according to (2.2) as $\delta_{PWr} = (S_{PWr},\ )_{MW}$. Explicitly, one has

$$\delta_{PW1} x^a = \delta_{P1} x^a, \tag{4.16a}$$

$$\delta_{PW1} y_a = \delta_{P1} y_a, \tag{4.16b}$$

$$\delta_{PW1} \beta_i = \delta_{W1} \beta_i, \tag{4.16c}$$

$$\delta_{PW1} \gamma^i = \delta_{W1} \gamma^i, \tag{4.16d}$$

$$\delta_{PW1} \mathrm{B}_i = \delta_{W1} \mathrm{B}_i, \tag{4.16e}$$

$$\delta_{PW1} \Gamma^i = \delta_{W1} \Gamma^i, \tag{4.16f}$$

$$\delta_{PW2} x^a = \delta_{P2} x^a + u_i{}^a(x) \gamma^i, \tag{4.16g}$$

$$\delta_{PW2} y_a = \delta_{P2} y_a - \partial_a u_i{}^b(x) \gamma^i y_b + \partial_a \mu_i(x) \Gamma^i, \tag{4.16h}$$

$$\delta_{PW2} \beta_i = \delta_{W2} \beta_i - u_i{}^a(x) y_a, \tag{4.16i}$$

$$\delta_{PW2} \gamma^i = \delta_{W2} \gamma^i, \tag{4.16j}$$

$$\delta_{PW2} \mathrm{B}_i = \delta_{W2} \mathrm{B}_i - \mu_i(x), \tag{4.16k}$$

$$\delta_{PW2} \Gamma^i = \delta_{W2} \Gamma^i, \tag{4.16l}$$

where the variations δ_{Pr}, δ_{Wr} are given in (4.9), (3.9), respectively.

From the above analysis, it follows again that the Poisson-Weil sigma model can be framed in the AKSZ scheme of section 2.

5. The Hitchin-Weil sigma model

In this section, we illustrate the Hitchin-Weil sigma model, which is the main topic of this paper. We follow closely the AKSZ treatment of ref. [20]. This will lead us on one hand to realize that the underlying geometry of the model is Poisson-quasi-Nijenhuis rather than generalized complex, on the other it will give us useful indications about symmetry reduction in this context, to be discussed in detail in section 6.

The target space of the Hitchin sigma model is a twisted manifold, i.e. a manifold M equipped with a closed 3-form $H \in C^\infty(M, \wedge^3 T^*M)$,[2]

$$\partial_a H_{bcd} - \partial_b H_{acd} + \partial_c H_{abd} - \partial_d H_{abc} = 0 \tag{5.1}$$

The field content of the Hitchin sigma model consists of a degree 0 embedding $x \in C^\infty(T[1]\Sigma, M)$ and a degree 1 section $y \in C^\infty(T[1]\Sigma, x^*T^*[1]M)$ as for the Poisson sigma model. The BV odd symplectic form is

$$\Omega_{M,H} = \int_{T[1]\Sigma} \varrho \Big[\delta x^a \delta y_a - \frac{1}{2} H_{abc}(x) \delta x^a dx^b \delta x^c\Big]. \tag{5.2}$$

It is easy to check that $\Omega_{M,H}$ satisfies

$$\delta \Omega_{M,H} = 0 \tag{5.3}$$

on account of (5.1). The associated BV antibrackets are given by

$$(F,G)_{M,H} = \int_{T[1]\Sigma} \varrho \Big[\frac{\delta_r F}{\delta x^a}\frac{\delta_l G}{\delta y_a} - \frac{\delta_r F}{\delta y_a}\frac{\delta_l G}{\delta x^a} + H_{abc}(x)\frac{\delta_r F}{\delta y_a} dx^b \frac{\delta_l G}{\delta y_c}\Big], \tag{5.4}$$

for any two functionals F, G on field space. See again app. B for technical details.

The model is characterized by two action functionals

$$S_{H1} = \int_{T[1]\Sigma} \varrho\, y_a dx^a + 2\int_\Gamma x^{(0)*} H, \tag{5.5a}$$

$$S_{H2} = \int_{T[1]\Sigma} \varrho \Big[\frac{1}{2} P^{ab}(x) y_a y_b + J^a{}_b(x) y_a dx^b\Big] + \int_\Gamma x^{(0)*}\Phi. \tag{5.5b}$$

Here, Γ is a 3-fold such that $\partial\Gamma = \Sigma$ and $x^{(0)} : \Gamma \to M$ is an embedding such that $x^{(0)}|_\Sigma$ equals the lowest degree 0 component of the embedding superfield x (see app. A) and whose choice is immaterial. $P \in C^\infty(M, \wedge^2 TM)$, $J \in C^\infty(M, \mathrm{End}\, TM)$, $\Phi \in C^\infty(M, \wedge^3 T^*M)$, are respectively a 2-vector, an endomorphism and a closed 3-form

$$\partial_a \Phi_{bcd} - \partial_b \Phi_{acd} + \partial_c \Phi_{abd} - \partial_d \Phi_{abc} = 0. \tag{5.6}$$

[2] The sign convention of the H field used here is opposite to that employed in ref. [20].

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Further, the compatibility condition

$$J^a{}_c P^{cb} + J^b{}_c P^{ca} = 0 \tag{5.7}$$

holds. The tensors P, J and Φ together define an almost Poisson-quasi-Nijenhuis structure [44]. The version of the Hitchin model presented here is more general than that originally expounded in [20], where the 3-form Φ was assumed to be exact (cf. eq. (5.12) below).

A straightforward computation yields the BV brackets

$$(S_{H1}, S_{H1})_{M,H} = 0, \tag{5.8a}$$

$$(S_{H1}, S_{H2})_{M,H} = 0, \tag{5.8b}$$

$$\begin{aligned}(S_{H2}, S_{H2})_{M,H} = 2\int_{T[1]\Sigma} \varrho \Big[&-\frac{1}{6} A_H{}^{abc}(x) y_a y_b y_c \\ &+ \frac{1}{2} B_H{}^{ab}{}_c(x) y_a y_b dx^c - \frac{1}{2} C_H{}^a{}_{bc}(x) y_a dx^b dx^c \Big],\end{aligned} \tag{5.8c}$$

where the tensor $A_H \in C^\infty(M, \wedge^3 TM)$, $B_H \in C^\infty(M, \wedge^2 TM \otimes T^*M)$, $C_H \in C^\infty(M, TM \otimes \wedge^2 T^*M)$ are given by

$$A_H{}^{abc} = P^{ad}\partial_d P^{bc} + P^{bd}\partial_d P^{ca} + P^{cd}\partial_d P^{ab}, \tag{5.9a}$$

$$\begin{aligned}B_H{}^{ab}{}_c = &J^d{}_c \partial_d P^{ab} + P^{ad}(\partial_c J^b{}_d - \partial_d J^b{}_c) - P^{bd}(\partial_c J^a{}_d - \partial_d J^a{}_c) \\ &- \partial_c(J^a{}_d P^{db}) - P^{ad}P^{be}H_{cde},\end{aligned} \tag{5.9b}$$

$$\begin{aligned}C_H{}^a{}_{bc} = &J^d{}_b \partial_d J^a{}_c - J^d{}_c \partial_d J^a{}_b - J^a{}_d \partial_b J^d{}_c + J^a{}_d \partial_c J^d{}_b \\ &+ P^{ad}\Phi_{dbc} + J^d{}_b P^{ae} H_{cde} - J^d{}_c P^{ae} H_{bde}.\end{aligned} \tag{5.9c}$$

Therefore, the joined BV master equations

$$(S_{Hr}, S_{Hs})_{M,H} = 0, \qquad r,\ s = 1,\ 2, \tag{5.10}$$

are satisfied if and only if

$$A_H{}^{abc} = 0, \tag{5.11a}$$

$$B_H{}^{ab}{}_c = 0, \tag{5.11b}$$

$$C_H{}^a{}_{bc} = 0. \tag{5.11c}$$

Conditions (5.11) are satisfied when the almost Poisson-quasi-Nijenhuis structure (P, J, Φ) is an H-twisted Poisson-quasi-Nijenhuis structure. A more restrictive notion of Poisson-quasi-Nijenhuis manifold was introduced by Stiénon and Xu in [44] in the untwisted case $H = 0$ (see section 6 below). As appears, the target space geometry of the Hitchin model encoded in the BV master equations is twisted Poisson-quasi-Nijenhuis. This broadens the scope of our original work on this model [20]. (See also [51, 52] for an alternative approach).

Twisted generalized complex geometry is a special case of twisted Poisson-quasi-Nijenhuis geometry. For a generalized almost complex manifold, the 3-form Φ is exact, so that one has

$$\Phi_{abc} = \partial_a Q_{bc} + \partial_b Q_{ca} + \partial_c Q_{ab}, \tag{5.12}$$

for some $Q \in C^\infty(M, \wedge^2 T^*M)$. The compatibility conditions are (5.7) and

$$J^a{}_c J^c{}_b + P^{ac} Q_{cb} + \delta^a{}_b \;=\; 0, \tag{5.13a}$$

$$Q_{ac} J^c{}_b + Q_{bc} J^c{}_a \;=\; 0. \tag{5.13b}$$

The differential conditions (5.11) are necessary but not sufficient for the target space generalized almost complex structure to be Courant integrable. To have Courant integrability, one needs, besides (5.11), a further condition

$$D_{Habc} = 0 \tag{5.14}$$

where $D_H \in C^\infty(M, \wedge^3 T^*M)$ is a 3-form defined by

$$\begin{aligned} D_{Habc} &= J^d{}_a \Phi_{dbc} + J^d{}_b \Phi_{dca} + J^d{}_c \Phi_{dab} - \partial_a (Q_{bd} J^d{}_c) - \partial_b (Q_{cd} J^d{}_a) \\ &\quad - \partial_c (Q_{ad} J^d{}_b) + H_{abc} - J^d{}_a J^e{}_b H_{cde} - J^d{}_b J^e{}_c H_{ade} - J^d{}_c J^e{}_a H_{bde}. \end{aligned} \tag{5.15}$$

The Courant integrability conditions (5.11), (5.14) were first derived in [18] and in equivalent form in [20] before Poisson-quasi-Nijenhuis geometry was formulated in [44].

The BV variations associated with the actions S_{Hr} are defined according to (2.2) as $\delta_{Hr} = (S_{Hr},\ )_{M,H}$. Explicitly, one has

$$\delta_{H1} x^a = dx^a, \tag{5.16a}$$

$$\delta_{H1} y_a = dy_a, \tag{5.16b}$$

$$\delta_{H2} x^a = P^{ab}(x) y_b + J^a{}_b(x) dx^b, \tag{5.16c}$$

$$\begin{aligned} \delta_{H2} y_a &= \frac{1}{2}\partial_a P^{bc}(x) y_b y_c + (\partial_a J^b{}_c - \partial_c J^b{}_a - P^{bd} H_{dac})(x) y_b dx^c \\ &\quad + J^b{}_a(x) dy_b + \frac{1}{2}(\Phi_{abc} - J^d{}_c H_{abd} + J^d{}_b H_{acd})(x) dx^b dx^c \end{aligned} \tag{5.16d}$$

From the above analysis, it follows that the also the Hitchin sigma model can be framed in the AKSZ scheme of section 2.

One can couple the Hitchin and the Weil sigma models and obtain the Hitchin-Weil sigma model, as one did for the Poisson sigma model. The field space of Hitchin-Weil model is simply the Cartesian product of those of the Hitchin and Weil models. The BV odd symplectic form $\Omega_{MW,H}$ of the Hitchin-Weil model is correspondingly the sum of those of the Hitchin and Weil models, $\Omega_{MW,H} = \Omega_{M,H} + \Omega_W$. The BV antibrackets $(\ ,\ )_{MW,H}$ are simply the sum of the BV antibrackets $(\ ,\ )_{M,H}$ and $(\ ,\ )_W$ given by (5.4), (3.3).

The Hitchin-Weil model is characterized by two action functionals,

$$S_{HW1} = S_{H1} + S_{W1}, \tag{5.17a}$$

$$\begin{aligned} S_{HW2} = S_{H2} + S_{W2} + \int_{T[1]\Sigma} \varrho \Big[& i\beta_i d\gamma^i - i\mathrm{B}_i d\Gamma^i - u_i{}^a(x)\gamma^i y_a \\ & - (\tau_{ia} - i\partial_a \mu_i)(x)\gamma^i dx^a + \mu_i(x)\Gamma^i \Big], \end{aligned} \tag{5.17b}$$

where $u \in C^\infty(M, \mathfrak{g}^\vee \otimes TM)$, $\tau \in C^\infty(M, \mathfrak{g}^\vee \otimes T^*M)$ and $\mu \in C^\infty(M, \mathfrak{g}^\vee)$ are a $\mathfrak{g}^\vee$-valued vector field, a $\mathfrak{g}^\vee$-valued 1-form and a $\mathfrak{g}^\vee$-valued scalar on M, respectively. We note that the action S_{HW2} is intrinsically complex because of the factors i appearing in the third term. The structure of S_{H2} my seem arbitrary at this stage. Its eventual justification will be provided by the results that will entail.

The computation of the BV brackets of the S_{HWr} is lengthy but completely straightforward. The result is

$$(S_{HW1}, S_{HW1})_{MW,H} = 0, \tag{5.18a}$$

$$(S_{HW1}, S_{HW2})_{MW,H} = 0, \tag{5.18b}$$

$$\begin{aligned}(S_{HW2}, S_{HW2})_{MW,H} =& (S_{H2}, S_{H2})_M + (S_{W2}, S_{W2})_W \\ &+ 2\int_{T[1]\Sigma} \varrho \Big[\frac{1}{2}X_i{}^{ab}(x)\gamma^i y_a y_b + Y_i{}^a{}_b(x)\gamma^i y_a dx^b + \frac{1}{2}Z_{iab}(x)\gamma^i dx^a dx^b \\ &- \frac{1}{2}L_{ij}{}^a(x)\gamma^i\gamma^j y_a - \frac{1}{2}M_{ija}(x)\gamma^i\gamma^j dx^a + N_{ij}(x)\gamma^i\Gamma^j \\ &- R_{ij}(x)\gamma^i d\gamma^j - S_i{}^a(x)\Gamma^i y_a - T_{ia}(x)\Gamma^i dx^a + V_i{}^a(x)d\gamma^i y_a\Big],\end{aligned} \tag{5.18c}$$

where the BV antibrackets $(S_{H2}, S_{H2})_M$, $(S_{W2}, S_{W2})_W$ are given by (5.8c), (3.5c), respectively, and $X \in C^\infty(M, \mathfrak{g}^\vee \otimes \wedge^2 TM)$, $Y \in C^\infty(M, \mathfrak{g}^\vee \otimes \mathrm{End}\, TM)$, $Z \in C^\infty(M, \mathfrak{g}^\vee \otimes \wedge^2 T^*M)$, $L \in C^\infty(M, \wedge^2\mathfrak{g}^\vee \otimes TM)$, $M \in C^\infty(M, \wedge^2\mathfrak{g}^\vee \otimes T^*M)$, $N, R \in C^\infty(M, \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee)$, $S, V \in C^\infty(M, \mathfrak{g}^\vee \otimes TM)$, $T \in C^\infty(M, \mathfrak{g}^\vee \otimes T^*M)$ are given by

$$X_i{}^{ab} = u_i{}^c\partial_c P^{ab} - \partial_c u_i{}^a P^{cb} - \partial_c u_i{}^b P^{ac}, \tag{5.19a}$$
$$Y_i{}^a{}_b = u_i{}^c\partial_c J^a{}_b - \partial_c u_i{}^a J^c{}_b + \partial_b u_i{}^c J^a{}_c - P^{ac}\Upsilon_{icb}, \tag{5.19b}$$
$$Z_{iab} = u_i{}^c\Phi_{cab} - \partial_a\Xi_{ib} + \partial_b\Xi_{ia} + J^c{}_a\Upsilon_{icb} - J^c{}_b\Upsilon_{ica}, \tag{5.19c}$$
$$L_{ij}{}^a = u_i{}^b\partial_b u_j{}^a - u_j{}^b\partial_b u_i{}^a - f^k{}_{ij}u_k{}^a, \tag{5.19d}$$
$$\begin{aligned}M_{ija} =& \frac{1}{2}\Big[u_i{}^b\partial_b\tau_{ja} + \partial_a u_i{}^b\tau_{jb} - u_j{}^b\partial_b\tau_{ia} - \partial_a u_j{}^b\tau_{ib} - 2f^k{}_{ij}\tau_{ka} \\ &- u_j{}^b\Upsilon_{iba} + u_i{}^b\Upsilon_{jba} - i\partial_a(u_i{}^b\partial_b\mu_j - u_j{}^b\partial_b\mu_i - 2f^k{}_{ij}\mu_k)\Big],\end{aligned} \tag{5.19e}$$
$$N_{ij} = u_i{}^a\partial_a\mu_j - f^k{}_{ij}\mu_k, \tag{5.19f}$$
$$R_{ij} = \frac{1}{2}\Big[u_i{}^a\tau_{ja} + u_j{}^a\tau_{ia} - i(u_i{}^a\partial_a\mu_j + u_j{}^a\partial_a\mu_i)\Big], \tag{5.19g}$$
$$S_i{}^a = u_i{}^a + P^{ab}\partial_b\mu_i, \tag{5.19h}$$
$$T_{ia} = \tau_{ia} - J^b{}_a\partial_b\mu_i, \tag{5.19i}$$
$$V_i{}^a = J^a{}_b u_i{}^b + P^{ab}(\tau_{ib} - i\partial_b\mu_i) - iu_i{}^a, \tag{5.19j}$$

where $\Xi \in C^\infty(M, \mathfrak{g}^\vee \otimes T^*M)$, $\Upsilon \in C^\infty(M, \mathfrak{g}^\vee \otimes \wedge^2 T^*M)$ are given by

$$\Xi_{ia} = i(\delta^b{}_a - iJ^b{}_a)(\tau_{ib} - i\partial_b\mu_i), \tag{5.19k}$$
$$\Upsilon_{iab} = \partial_a\tau_{ib} - \partial_b\tau_{ia} - u_i{}^c H_{cab}. \tag{5.19l}$$

The joined BV master equations

$$(S_{HWr}, S_{HWs})_{MW,H} = 0, \qquad r,\ s = 1,\ 2, \tag{5.20}$$

are satisfied if and only if (5.11), the conditions

$$N_{ij} = 0, \tag{5.21a}$$
$$S_i{}^a = 0, \tag{5.21b}$$
$$T_{ia} = 0, \tag{5.21c}$$

and (3.8) are simultaneously fulfilled. Indeed, it is not difficult to check that, when u_i and τ_i are given by (5.21b) and (5.21c), respectively, one has

$$X_i{}^{ab} = A_H{}^{cab}\partial_c\mu_i, \tag{5.22a}$$
$$Y_i{}^a{}_b = -B_H{}^{ca}{}_b\partial_c\mu_i, \tag{5.22b}$$
$$Z_{iab} = C_H{}^c{}_{ab}\partial_c\mu_i \tag{5.22c}$$
$$L_{ij}{}^a = A_H{}^{abc}\partial_b\mu_i\partial_c\mu_j - P^{ab}\partial_b N_{ij}, \tag{5.22d}$$
$$M_{ija} = -B_H{}^{bc}{}_a\partial_b\mu_i\partial_c\mu_j - i(\delta^b{}_a + iJ^b{}_a)\partial_b N_{ij}, \tag{5.22e}$$
$$R_{ij} = 0, \tag{5.22f}$$
$$V_i{}^a = 0. \tag{5.22g}$$

The geometrical interpretation of conditions will be analyzed later in section 6. We anticipate that the geometry they describe is closely related to but more general than that of reduction of generalized complex and Kaehler manifolds under a group action recently developed by Lin and Tolman in [40, 40] and may suggest a viable framework for reduction of Poisson-quasi-Nijenhuis manifolds.

In the formulation of refs. [40, 40], generalized complex geometry being concerned, (5.12)–(5.14) hold true. In addition to (5.11), (5.21) and (3.8), it is further assumed that

$$\Upsilon_{ia} = 0, \tag{5.23}$$

where Υ is given by (5.19l). All the tensors appearing in (5.22) continue of course to vanish, but one also has a further relation, which pairs with (5.22g),

$$W_{ia} = 0, \tag{5.24}$$

where $W \in C^\infty(M, \mathfrak{g}^\vee \otimes T^*M)$ is given by

$$W_{ia} = Q_{ab}u_i{}^b - J^b{}_a(\tau_{ib} - i\partial_b\mu_i) - i(\tau_{ia} - i\partial_a\mu_i). \tag{5.25}$$

These conditions plus other regularity conditions are sufficient to ensure the existence of a reduction of the relevant generalized complex manifold.

The BV variations associated with the actions S_{HWr} are defined as usual according to (2.2) as $\delta_{HWr} = (S_{HWr},\ )_{MW,H}$. Explicitly,

$$\delta_{HW1}x^a = \delta_{H1}x^a, \tag{5.26a}$$

$$\delta_{HW1}y_a = \delta_{H1}y_a, \tag{5.26b}$$

$$\delta_{HW1}\beta_i = \delta_{W1}\beta_i, \tag{5.26c}$$

$$\delta_{HW1}\gamma^i = \delta_{W1}\gamma^i, \tag{5.26d}$$

$$\delta_{HW1}\mathrm{B}_i = \delta_{W1}\mathrm{B}_i, \tag{5.26e}$$

$$\delta_{HW1}\Gamma^i = \delta_{W1}\Gamma^i, \tag{5.26f}$$

$$\delta_{HW2}x^a = \delta_{H2}x^a + u_i{}^a(x)\gamma^i, \tag{5.26g}$$

$$\begin{aligned}\delta_{HW2}y_a =& \delta_{H2}y_a - \partial_a u_i{}^b(x)\gamma^i y_b - (\tau_{ia} - i\partial_a\mu_i)(x)d\gamma^i \\ &- (\partial_a\tau_{ib} - \partial_b\tau_{ia} - u_i{}^c H_{cab})(x)\gamma^i dx^b + \partial_a\mu_i(x)\Gamma^i,\end{aligned} \tag{5.26h}$$

$$\delta_{HW2}\beta_i = \delta_{W2}\beta_i + id\beta_i - u_i{}^a(x)y_a - (\tau_{ia} - i\partial_a\mu_i)(x)dx^a, \tag{5.26i}$$

$$\delta_{HW2}\gamma^i = \delta_{W2}\gamma^i + id\gamma^i, \tag{5.26j}$$

$$\delta_{HW2}\mathrm{B}_i = \delta_{W2}\mathrm{B}_i + id\mathrm{B}_i - \mu_i(x), \tag{5.26k}$$

$$\delta_{HW2}\Gamma^i = \delta_{W2}\Gamma^i + id\Gamma^i, \tag{5.26l}$$

where the variations δ_{Hr}, δ_{Wr} are given in (5.16), (3.9), respectively.

From the above analysis, it follows that also the Hitchin-Weil sigma model can be framed in the AKSZ scheme of section 2.

b transformation is the basic symmetry of generalized complex geometry. Though originally discovered in this context, b transformation can be straightforwardly generalized to twisted Poisson-quasi-Nijenhuis geometry. For a thorough analysis of the significance of b transformation, the reader is referred to [6].

b transformation is parameterized by a 2-form $b \in C^\infty(M, \wedge^2 T^*M)$. It acts in the 3-form H by shifting it by $d_M b$:

$$H'_{abc} = H_{abc} + \partial_a b_{bc} + \partial_b b_{ca} + \partial_c b_{ab}. \tag{5.27}$$

It acts also on the tensors P, J and Φ defining an almost Poisson-quasi-Nijenhuis structure by setting

$$P'^{ab} = P^{ab}, \tag{5.28a}$$

$$J'^a{}_b = J^a{}_b - P^{ac}b_{cb}, \tag{5.28b}$$

$$\Phi'_{abc} = \Phi_{abc} + \partial_a\phi_{bc} + \partial_b\phi_{ca} + \partial_c\phi_{ab}, \tag{5.28c}$$

where ϕ_{ab} is given by

$$\phi_{ab} = b_{ac}J^c{}_b - b_{bc}J^c{}_a + P^{cd}b_{ca}b_{db}. \tag{5.28d}$$

It is immediate to see that the BV odd symplectic form $\Omega_{M,H}$ given in (5.2) is not invariant under b transformation [20]. To render it invariant, it is necessary to make b transformation act also on the sigma model fields as

$$x'^a = x^a, \tag{5.29a}$$

$$y'_a = y_a + b_{ab}(x)dx^b. \tag{5.29b}$$

One then has

$$\Omega'_{M,H} = \Omega_{M,H}, \tag{5.30}$$

as required. It is straightforward to verify that the Hitchin action functionals S_{Hr} are also both invariant under b transformation,

$$S'_{Hr} = S_{Hr}, \qquad r = 1,\ 2. \tag{5.31}$$

This shows that b transformation is a duality symmetry of the Hitchin model [20].

b transformation can be rendered a symmetry of the Hitchin-Weil model if we stipulate further that the tensors u_i, τ_i and μ_i transform as

$$u'_i{}^a = u_i{}^a, \tag{5.32a}$$

$$\tau'_{ia} = \tau_{ia} + b_{ab} u_i{}^b, \tag{5.32b}$$

$$\mu'_i = \mu_i. \tag{5.32c}$$

Upon doing this, it is readily seen that the Hitchin-Weil action functionals S_{HWr} are also both invariant under b transformation,

$$S'_{HWr} = S_{HWr}, \qquad r = 1,\ 2. \tag{5.33}$$

As we shall see, b symmetry plays an important role also in the analysis of reduction given in the next section.

6. Geometrical interpretation

Let M be a manifold. An almost Poisson structure on M is an element $P \in C^\infty(M, \wedge^2 TM)$. An almost Poisson structure P is a Poisson structure if

$$[P, P] = 0, \tag{6.1}$$

where $[\ ,\ ]$ denotes the Schoutens-Nijenhius brackets. (More explicitly, $[P, P] \in C^\infty(M, \wedge^3 TM)$ is given by the right hand side of (6.8a) below). (6.1) is nothing but (4.8) expressed in coordinate free form. As is well known, when a Poisson structure P on M is given, one can define Poisson brackets on $C^\infty(M)$ in standard fashion.

Assume now that the our Poisson manifold (M, P) carries the action of a connected Lie group G with Lie algebra $\mathfrak{g}$ represented infinitesimally by the $\mathfrak{g}^\vee$-valued vector field $u \in C^\infty(M, \mathfrak{g}^\vee \otimes TM)$. The action is said Hamiltonian, if there exist a $\mathfrak{g}^\vee$-valued scalar $\mu \in C^\infty(M, \mathfrak{g}^\vee)$, called the moment map, such that[3]

$$u_i = -P d_M \mu_i, \tag{6.2a}$$

$$\{\mu_i, \mu_j\} = f^k{}_{ij} \mu_k. \tag{6.2b}$$

[3] Here and below, we view P equivalently as a section of $\mathrm{Hom}(T^*M, TM)$.

These are precisely conditions (4.14) written in intrinsic notation. (6.2) automatically implies that

$$l_{u_i}P = 0, \tag{6.3a}$$

$$l_{u_i}u_j - f^k{}_{ij}u_k = 0, \tag{6.3b}$$

as required by the invariance of P and the equivariance of u. These are relations (4.15) upon taking (4.8), (4.14) into account written again in intrinsic notation.

A classic result of Marsden and Ratiu [53] (see also [54]) ensures that, under these conditions, if $a \in \mathfrak{g}^\vee$ with coadjoint orbit $\mathcal{O}_a$ and $\mu^{-1}(\mathcal{O}_a)$ is a submanifold of M on which G acts freely and properly, then the quotient $M_a = \mu^{-1}(\mathcal{O}_a)/G$ inherits a Poisson structure P_a. Thus, the Poisson-Weil model described in section 4 encodes Poisson reduction.

Next, we want to analyze the extent to which the above standard Poisson reduction framework extends to Poisson-quasi-Nijenhuis structures. To the best of our knowledge, no such reduction scheme has been been developed so far. However, since, as shown above, Poisson reduction is encoded in the Poisson-Weil model, it is reasonable to expect that Poisson-quasi-Nijenhuis reduction may be encoded in the Hitchin-Weil model expounded in section 5.

Poisson-quasi-Nijenhuis structures were first introduced by Stiénon and Xu in [44], who, in turn, were inspired by earlier work by Magri e Morosi [55]. The authors of [44] considered only the untwisted case, but their analysis can be extended to the twisted case directly.

A manifold M is called twisted if it is equipped with a closed 3-form $H \in C^\infty(M, \wedge^3 T^*M)$

$$d_M H = 0. \tag{6.4}$$

Henceforth, we assume that M is twisted.

An almost Poisson-quasi-Nijenhuis structure on M is a triple (J, P, Φ), where $P \in C^\infty(M, \wedge^2 TM)$, $J \in C^\infty(M, \mathrm{End}\, TM)$, $\Phi \in C^\infty(M, \wedge^3 T^*M)$ with

$$d_M \Phi = 0, \tag{6.5}$$

(cf. eq. (5.6)) and satisfying the compatibility condition

$$JP - PJ^t = 0 \tag{6.6}$$

(cf. eq. (5.7)). An almost Poisson-quasi-Nijenhuis structure (J, P, Φ) on M is an H twisted Poisson-quasi-Nijenhuis structure if

$$A_H = 0, \tag{6.7a}$$

$$B_H = 0, \tag{6.7b}$$

$$C_H = 0, \tag{6.7c}$$

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where the tensor $A_H \in C^\infty(M, \wedge^3 TM)$, $B_H \in C^\infty(M, \wedge^2 TM \otimes T^*M)$, $C_H \in C^\infty(M, TM \otimes \wedge^2 T^*M)$ are defined by

$$A_H(\alpha,\beta) = [P\alpha, P\beta] - P\{\alpha,\beta\}_P, \tag{6.8a}$$

$$B_H(\alpha,\beta) = \{\alpha, J^t\beta\}_P - \{\beta, J^t\alpha\}_P - \{\alpha,\beta\}_{PJ^t} - J^t\{\alpha,\beta\}_P + i_{P\alpha} i_{P\beta} H, \tag{6.8b}$$

$$\begin{aligned} C_H(X,Y) = {} & [JX, JY] - J\big([JX,Y] - [JY,X] - J[X,Y]\big) \\ & - P\big(i_X i_Y \Phi - i_{JX} i_Y H + i_{JY} i_X H\big), \end{aligned} \tag{6.8c}$$

where $\alpha, \beta \in C^\infty(M, T^*M)$ and $X, Y \in C^\infty(M, TM)$,

$$\{\alpha,\beta\}_K = l_{K\alpha}\beta - l_{K\beta}\alpha - \frac{1}{2} d_M(i_{K\alpha}\beta - i_{K\beta}\alpha), \tag{6.8d}$$

for $K \in C^\infty(M, \wedge^2 TM)$, and l and i denote Lie derivation and contraction, respectively. It is straightforward to check that the local coordinate expressions of A_H, B_H, C_H are precisely those given by eq. (5.9), justifying the claim previously made about the underlying geometry of the Hitchin model.

In [44], a further condition is added (in the $H = 0$ case). The 3-form Φ is required to satisfy the condition

$$d i_J \Phi = 0, \qquad (H = 0). \tag{6.9}$$

To understand the reason of this condition, we recall the following result proven in [44]. The conditions (6.7) together are equivalent to: 1) $(T^*M, \{,\}, P)$ being a Lie algebroid; 2) $d_J = [i_J, d]$ being a degree 1 derivation of the associated Gerstenhaber algebra $(C^\infty(M, \wedge^* T^*M), \wedge, [.,.])$; 3) the relation $d_J{}^2 = [\Phi, .]$. These three properties together with (6.9) render $(T^*M, \{,\}, P, d_J, \Phi)$ a quasi Lie bialgebroid. Thus, an untwisted Poisson-quasi Nijenhuis structure on M, in the more restricted sense used here, is tantamount of a quasi Lie bialgebroid structure on T^*M. The condition (6.9) is added, among other things, because the relation $d_J{}^2 = [\Phi, .]$ requires as a consistency condition that $[d_J\Phi, .] = 0$ and, as $d\Phi = 0$, (6.9) is sufficient for this to hold. This indicates that the three conditions (6.7) imply (6.9) or some mild generalization of it. As we have seen, (6.9) does not follow from our BV analysis. The classical BV master equation yields the conditions which the target space geometry must satisfy for the welldefinedness of the model, but of course it does not yield the consistency conditions which these imply. When $H \neq 0$, the generalization of (6.9) is not known. Inspection of the condition (5.14) holding in generalized complex geometry suggests the following condition

$$d\Big[i_J\Phi + H + \frac{1}{2}(i_{J^2} - i_J i_J)H\Big] = 0. \tag{6.10}$$

This is however a conjecture for the time being. Further investigation on this point seems necessary.

Poisson-quasi-Nijenhuis geometry is covariant not only under diffeomorphism symmetry but also under b transformation symmetry. For $b \in C^\infty(M, \wedge^2 T^*M)$, the b-transform of the 3-form H is

$$H' = H + d_M b, \tag{6.11}$$

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(cf. eq. (5.27)). The b transform of an almost Poisson-quasi-Nijenhuis structure (P, J, Φ) is given by

$$P' = P, \tag{6.12a}$$

$$J' = J - Pb, \tag{6.12b}$$

$$\Phi' = \Phi + d_M(i_J b - bPb), \tag{6.12c}$$

(cf. eq. (5.28)). It is straightforward though lengthy to verify that (P, J, Φ) is an H twisted Poisson-quasi-Nijenhuis structure, then (P', J', Φ') is H' twisted Poisson-quasi-Nijenhuis structure.

Assume now that the our Poisson-quasi-Nijenhuis manifold (M, P, J, Φ) carries the action of a connected Lie group G with Lie algebra $\mathfrak{g}$. Intuitively, since the relevant vector bundle in Poisson-quasi-Nijenhuis is $TM \oplus T^*M$ rather than TM, as in generalized complex geometry, we expect that the G action is represented at the infinitesimal level not only by a $\mathfrak{g}^\vee$-valued vector field $u \in C^\infty(M, \mathfrak{g}^\vee \otimes TM)$, as above, but also by a $\mathfrak{g}^\vee$-valued 1-form $\tau \in C^\infty(M, \mathfrak{g}^\vee \otimes T^*M)$, which we name moment 1-form in compliance with common usage [32–35, 40, 41]. We call the G action Hamiltonian, if there exist a $\mathfrak{g}^\vee$-valued scalar $\mu \in C^\infty(M, \mathfrak{g}^\vee)$, called the moment map, such that

$$u_i = -P d_M \mu_i, \tag{6.13a}$$

$$\tau_i = J^t d_M \mu_i, \tag{6.13b}$$

$$\{\mu_i, \mu_j\} = f^k{}_{ij} \mu_k. \tag{6.13c}$$

These are precisely conditions (5.21) written in intrinsic notation. They generalize (6.2) in obvious fashion. u_i and τ_i organize in the following section $A \in C^\infty(M, \mathfrak{g}^\vee \otimes (TM \oplus T^*M))$

$$A_i = u_i + \tau_i \tag{6.14}$$

satisfying the relations

$$\langle A_i, A_j \rangle = 0, \tag{6.15}$$

$$[\![A_i, A_j]\!]_H - f^k{}_{ij} A_k = 0, \tag{6.16}$$

where $\langle \cdot, \cdot \rangle$ and $[\![\cdot, \cdot]\!]_H$ are the metric and the H twisted Courant brackets of $TM \oplus T^*M$. A acts on $C^\infty(M, TM \oplus T^*M)$ by the Courant adjoint action

$$A_i \cdot X = [\![A_i, X]\!]_H, \qquad X \in C^\infty(M, TM \oplus T^*M). \tag{6.17}$$

This defines a trivially extended $\mathfrak{g}$ action on $TM \oplus T^*M$ preserving its Courant algebroid structure [32, 33]. This action should integrate to G action in order reduction to be preformed.

When (6.7), (6.13) hold, one has

$$l_{u_i} P = 0, \tag{6.18a}$$

$$l_{u_i} J - P \Upsilon_i = 0 \tag{6.18b}$$

$$i_{u_i} \Phi - d_M \Xi_i + i_J \Upsilon_i = 0 \tag{6.18c}$$

$$l_{u_i} u_j - f^k{}_{ij} u_k = 0, \tag{6.18d}$$

$$l_{u_i} \tau_j - f^k{}_{ij} \tau_k - i_{u_j} \Upsilon_i = 0, \tag{6.18e}$$

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where $\Xi \in C^\infty(M, \mathfrak{g}^\vee \otimes T^*M)$, $\Upsilon \in C^\infty(M, \mathfrak{g}^\vee \otimes \wedge^2 T^*M)$ are given by

$$\Xi_i = (1 + i_J i_J) d_M \mu_i, \tag{6.18f}$$

$$\Upsilon_i = d_M \tau_i - i_{u_i} H. \tag{6.18g}$$

These are relations (5.22) upon taking (5.11), (5.21) into account written again in intrinsic notation. They generalize (6.3) in a rather non trivial way. We see that H is not invariant and that, while P is invariant, J, Φ fail to be so. Similarly, while u is equivariant, τ is not. In all cases, the obstruction is given by the 2-form Υ.

In the presence of a G action on M, the above geometric framework is covariant under b transformation provided this acts also on u, τ and μ as

$$u'_i = u_i, \tag{6.19a}$$

$$\tau'_i = \tau_i - i_{u_i} b, \tag{6.19b}$$

$$\mu'_i = \mu_i, \tag{6.19c}$$

(cf. eq. (5.32)). From these relations and from (6.18), one realizes immediately that the failure of H, J, Φ to be invariant and, similarly, of τ to be equivariant has the form of an infinitesimal b transform with $b = \Upsilon_i$ for given i. That this comes about is hardly surprising, given the b symmetry of the Hitchin-Weil model, from which (6.18) were obtained. It reflects also the fact that the symmetry of the geometry considered here is larger than the diffeomorphism one and contains also b transformation, as in generalized complex geometry. Note that the cohomology class $[H] \in H^3(M, \mathbb{R})$ and the b symmetry equivalence class of (P, J, Φ) are both invariant.

The natural question arises about whether it is possible to make all the Υ_i vanish by means of a single b transform. It is easy to see that, to this end, it is sufficient that the b field solves the equation

$$l_{u_i} b = \Upsilon_i, \tag{6.20}$$

for all i. It can be shown that such a b field exists if G is a compact Lie group and if the Lie algebra action (6.17) integrates to a G action [32, 36]. Alternatively, one may impose the condition

$$\Upsilon_i = 0, \tag{6.21}$$

by hand. This, however, is not yielded by the formalism in natural fashion.

It is natural to wonder whether the above provides a viable framework for the reduction of Poisson-quasi-Nijenhuis structures. We have no answer as yet, since we have no mathematical literature with which to compare our results. It is however useful to that end to examine what is known about reduction in generalized complex geometry.

A generalized almost complex structure $\mathcal{J}$ is a section of $C^\infty(\mathrm{End}(TM \oplus T^*M))$, which is an isometry of the natural Courant metric $\langle\ ,\ \rangle$ of $TM \oplus T^*M$ and satisfies

$$\mathcal{J}^2 = -1 \tag{6.22}$$

[6]. The generalized almost complex structure $\mathcal{J}$ is called a generalized complex structure if its $+i$ eigenbundles $L_{\mathcal{J}}$ of $\mathcal{J}$ is involutive with respect to the Courant brackets $[\![\cdot,\cdot]\!]_H$ of $TM \oplus T^*M$ [6].[4]

It is often convenient to write a generalized almost complex structure $\mathcal{J}$ in the block form

$$\mathcal{J} = \begin{pmatrix} J & P \\ Q & -J^t \end{pmatrix}, \tag{6.23}$$

where $P \in C^\infty(M, \wedge^2 TM)$, $J \in C^\infty(M, \mathrm{End}TM)$, $Q \in C^\infty(M, \wedge^2 T^*M)$. It is easily checked that the triple (P, J, Φ), where

$$\Phi = d_M Q, \tag{6.24}$$

is an almost Poisson-quasi-Nijenhuis structure satisfying besides (6.6) two more algebraic conditions following from (6.22) and corresponding to eq. (5.13). If $\mathcal{J}$ is a generalized complex structure, then (P, J, Φ) is a Poisson-quasi-Nijenhuis structure satisfying besides (6.7) an extra differential condition following from Courant involutivity of $L_{\mathcal{J}}$ and corresponding to eq. (5.14),

$$\begin{aligned} &\Phi(JX, Y, Z) + \Phi(JY, Z, X) + \Phi(JZ, X, Y) - d_M(QJ)(X, Y, Z) \\ &\quad + H(X, Y, Z) - H(JX, JY, Z) - H(JY, JZ, X) - H(JZ, JX, Y) = 0, \end{aligned} \tag{6.25}$$

with $X, Y, Z \in C^\infty(M, TM)$.

Assume now that our generalized complex manifold $(M, \mathcal{J})$ carries the the action of a compact Lie group G with Lie algebra $\mathfrak{g}$ represented infinitesimally by the vector fields u_i. Following Lin and Tolman [40, 41] (see also [34]), we define a generalized moment map to be an element $Z \in C^\infty(M, \mathfrak{g}^\vee \otimes (TM \oplus T^*M) \otimes \mathbb{C})$ of the form

$$Z_i = u_i + \tau_i - i d_M \mu_i \tag{6.26}$$

such that

$$\mathcal{J} Z_i = i Z_i \tag{6.27}$$

and that (6.13c) holds. It is easy to see that (6.27) implies (6.13a), (6.13b) and summarizes in intrinsic form (5.22g), (5.24).

Let us assume that (6.21) holds. (6.21) is just (5.23). From (6.18a)–(6.18c) and (6.21), it follows that H and P, J, Q and, so, $\mathcal{J}$ are all invariant. Similarly, (6.18e) and (6.21) imply that τ is equivariant. According to the authors of [40, 41], under these conditions, if, for $a \in \mathfrak{g}^\vee$ with coadjoint orbit $\mathcal{O}_a$ and if $\mu^{-1}(\mathcal{O}_a)$ is a submanifold of M on which G acts freely, then the quotient $M_a = \mu^{-1}(\mathcal{O}_a)/G$ inherits a generalized complex structure $\mathcal{J}_a$.

The above analysis shows that the reduction scheme of Lin and Tolman is a particular case of the one worked out in this paper. It seems therefore to point to a reduction

[4] The $\pm i$ eigenbundles of $\mathcal{J}$ are complex and, thus, their analysis requires complexifying $TM \oplus T^*M$ leading to $(TM \oplus T^*M) \otimes \mathbb{C}$.

framework far more general than that considered by Lin and Tolman. One one hand, it may apply to Poisson-quasi-Nijenhuis structures, which are more general than generalized complex ones. On the other, strict invariance may not be necessary at the end and the weaker conditions (6.18a)–(6.18c) may suffice.

7. Discussion

In sects. 4, 5, we have argued that the Poisson-Weil and Hitchin-Weil sigma models encode the symmetry reduction of the Poisson and Hitchin sigma models, respectively. In a sense, coupling to the Weil model should perform the same type of function as gauging and may be considered to be a gauging in a sense, though, strictly speaking, there is no gauge field that interacts with the ungauged sigma model fields.

The sigma models studied in this paper cannot be considered fully fledged quantum field theories as long as gauge fixing is not carried out, since, in the absence of gauge fixing, the kinetic terms of the fields are ill defined. Fixing the gauge requires restricting the fields on a suitable functional submanifold $\mathfrak{L}$ in field space, that is Lagrangian with respect to the BV odd symplectic form [23 − 25]. The restriction results in certain relations among the fields. Formal arguments, based on the BV master equation, indicate that the resulting gauge fixed field theory is independent at the quantum level from the choice of $\mathfrak{L}$ as long as the choices considered can be continuously deformed one into another. Unfortunately, fixing the gauge is usually a technically very hard problem [25, 26].

We have seen that symmetry reduction of a Poisson or a generalized complex manifold requires the choice of some element $a \in \mathfrak{g}^\vee$. The reduced manifold is then the quotient $M_a = \mu^{-1}(\mathcal{O}_a)/G$, where $\mathcal{O}_a$ is the coadjoint orbit of a. However, there is no trace of such a choice in the models we described. It is likely that a enters in some way in the definition of the functional Lagrangian submanifold $\mathfrak{L}$ involved in gauge fixing in the Weil sector of the sigma model. However, at the moment, this is only a speculation. Clearly, much work remains to be done to reach a better understanding of these matters.

Acknowledgments

We thank J. Louis for organizing the Workshop on ”Generalized Geometry and Flux Compactifications” held in DESY from February 19th through March 1st 2007, during which this paper was first conceived. We thank J.-P. Ortega, H. Bursztyn, B. Uribe. T. Strobl and F. Bonechi for sharing with me their insight in Hamiltonian reduction theory and sigma models. We thank also the referee for useful criticism concerning the proper interpretation of the AKSZ theory. Finally, we thank The Erwin Schroedinger Institute for Mathematical Physics for the kind hospitality offered to us in August 2007 as a participant to the ESI program “Poisson sigma models, Lie groupoids, deformations and higher analogues”.

A. De Rham superfields

In general, the fields of a 2-dimensional field theory are differential forms on a oriented closed 2-dimensional manifold Σ. They can be viewed as elements of the space $C^\infty(T[1]\Sigma)$

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of functions on the Grassmann degree 1 tangent bundle $T[1]\Sigma$ of Σ, which we shall call de Rham superfields. More explicitly, we associate with the coordinates z^α of Σ Grassmann odd partners ζ^α with

$$\deg z^\alpha = 0, \qquad \deg \zeta^\alpha = 1. \tag{A.1}$$

$T[1]\Sigma$ is endowed with a natural differential d defined by

$$dz^\alpha = \zeta^\alpha, \qquad d\zeta^\alpha = 0. \tag{A.2}$$

A generic de Rham superfield $\psi(z,\zeta)$ is a triplet formed by a 0–, 1–, 2-form field $\psi^{(0)}(z)$, $\psi^{(1)}{}_\alpha(z)$, $\psi^{(2)}{}_{\alpha\beta}(z)$ organized as

$$\psi(z,\zeta) = \psi^{(0)}(z) + \zeta^\alpha\psi^{(1)}{}_\alpha(z) + \frac{1}{2}\zeta^\alpha\zeta^\beta\psi^{(2)}{}_{\alpha\beta}(z). \tag{A.3}$$

The forms $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ are called the components of ψ. Note that, in this formalism, the exterior differential of Σ can be identified with the operator

$$d = \zeta^\alpha \partial/\partial z^\alpha. \tag{A.4}$$

The coordinate invariant integration measure of $T[1]\Sigma$ is

$$\varrho = dz^1 dz^2 d\zeta^1 d\zeta^2. \tag{A.5}$$

Any de Rham superfield ψ can be integrated on $T[1]\Sigma$ according to the prescription

$$\int_{T[1]\Sigma} \varrho\, \psi = \int_\Sigma \frac{1}{2}\, dz^\alpha dz^\beta \psi^{(2)}{}_{\alpha\beta}(z). \tag{A.6}$$

By Stokes' theorem,

$$\int_{T[1]\Sigma} \varrho\, d\psi = 0. \tag{A.7}$$

It is possible to define functional derivatives of functionals of de Rham superfields. Let ψ be a de Rham superfield and let $F(\psi)$ be a functional of ψ. We define the left/right functional derivative superfields $\delta_{l,r}F(\psi)/\delta\psi$ as follows. Let σ be a superfield of the same properties as ψ. Then,

$$\frac{d}{dt}F(\psi + t\sigma)\Big|_{t=0} = \int_{T[1]\Sigma} \varrho\, \sigma \frac{\delta_l F(\psi)}{\delta\psi} = \int_{T[1]\Sigma} \varrho\, \frac{\delta_r F(\psi)}{\delta\psi}\sigma. \tag{A.8}$$

In the applications below, the components of the relevant de Rham superfields carry, besides the form degree, also a ghost degree. We shall limit ourselves to homogeneous superfields. A de Rham superfield ψ is said homogeneous if the sum of the form and ghost degree is the same for all its components $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$ of ψ. The common value of that sum is called the (total) degree $\deg\psi$ of ψ. It is easy to see that the differential operator d and the integration operator $\int_{T[1]\Sigma}\varrho$ carry degree 1 and -2, respectively. Also, if $F(\psi)$ is a functional of a superfield ψ, then $\deg \delta_{l,r}F(\psi)/\delta\psi = \deg F - \deg\psi + 2$.

B. The functional derivation $\delta/\delta x^a$

Since, for given $x \in C^\infty(T[1]\Sigma, M)$, one has $y \in C^\infty(T[1]\Sigma, x^*T^*[1]M)$, it is not possible to vary x keeping y fixed. In fact, the condition $\delta y = 0$ is not covariant, as is easy to see, and, so, it cannot be consistently imposed. This poses a technical problem for the computation of the functional derivatives $\delta F/\delta x^a$, when F explicitly depends on y. The difficulty is solved by picking a connection Γ of M and requiring that

$$\delta_{\mathrm{cov}} y_a = \delta y_a - \Gamma^b{}_{ca}(x)\delta x^c y_b = 0, \tag{B.1}$$

under variation of x. It is convenient to take Γ torsionless. One then computes $\delta_{\mathrm{cov}} F/\delta x^a$ by varying both x and y with δy given by (B.1). The result depends of course on the choice Γ. However, in all the relevant calculations, Γ drops out at the end, reflecting the intrinsic covariance of the theory.

The BV brackets (4.3), (5.4) are to be computed by replacing $\delta/\delta x^a$ by $\delta_{\mathrm{cov}}/\delta x^a$ throughout. It can be checked that the result does not depend on Γ. Similarly, if S_t is a BV master action, then the BV variations, obtained from

$$\delta_t x^a = (S_t, x^a), \tag{B.2a}$$

$$\delta_t y_a - \Gamma^b{}_{ca}(x)\delta_t x^c y_b = (S_t, y_a), \tag{B.2b}$$

also do not depend on Γ.

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